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Abstract: Macroscopic entropy of an extremal black hole is expected to be determined completely by its near horizon geometry. Thus two black holes with identical near horizon geometries should have identical macroscopic entropy, and the expected equality between macroscopic and microscopic entropies will then imply that they have identical degeneracies of microstates. An apparent counterexample is provided by the 4D-5D lift relating BMPV black hole to a four dimensional black hole. The two black holes have identical near horizon geometries but different microscopic spectrum. We suggest that this discrepancy can be accounted for by black hole hair — degrees of freedom living outside the horizon and contributing to the degeneracies. We identify these degrees of freedom for both the four and the five dimensional black holes and show that after their contributions are removed from the microscopic degeneracies of the respective systems, the result for the four and five dimensional black holes match exactly.

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JHEP07(2009)091

Contents

1 Introducion and summary

Since the Bekenstein-Hawking entropy of a black hole is proportional to the area of the event horizon of the black hole [1–3] one expects that the horizon of the black hole contains the key to understanding the black hole microstates. Wald's modification of the Bekenstein-Hawking formula in higher derivative theories of gravity [4–7] deviates from the area law, but nevertheless expresses the black hole entropy in terms of the horizon data. The situation becomes even better in the extremal limit where an infinite throat separates the horizon from the rest of the black hole space-time and the near horizon configuration can be regarded as a fully consistent solution to the field equations [8–12]. The classical Wald entropy can be related to the value of the classical Lagrangian density evaluated in this near horizon geometry [13]. This leads one to expect that we should be able to define a macroscopic quantity, computed from quantum string theory in the near horizon geometry, that captures complete information about the microscopic degeneracies of the corresponding black hole. Quantum entropy function is such a proposal relating the microscopic degeneracies of extremal black holes to an appropriate partition function of quantum gravity in the near horizon geometry of the black hole [14, 15] (see also [16, 17]).

Irrespective of any specific proposal, if the postulate that the microscopic degeneracy of an extremal black hole can be related to some computation in the near horizon geometry is correct, then this leads to an immediate consequence: two black holes with identical near horizon geometries will have identical degeneracies of microstates. There are some trivial counterexamples with straightforward resolutions. For example the near horizon geometry of an extremal black hole in flat space-time is independent of the asymptotic values of the moduli fields due to the attractor mechanism [9–11, 13, 18, 19], but the microscopic degeneracy of states, carrying the same quantum numbers as the black hole, jumps across the walls of marginal stability as we vary the asymptotic moduli [20–24]. The resolution of this puzzle is provided by the fact that for a given set of charges there are typically many classical solutions. One of these is a single centered black hole solution but the others contain multiple centers [24–31]. As we cross a wall of marginal stability some of these multi-centered solutions cease to exist and hence cause a jump in the total entropy. This precisely accounts for the jump in the total degeneracy across the walls of marginal stability, thereby showing that the degeneracy of states associated with a single centered black hole remains unchanged as we cross a wall of marginal stability. This suggests a natural modification of the original proposal: *string theory in the near horizon geometry captures information about the microscopic degeneracy of the single centered black holes only.* This is clearly natural from a physical perspective: the near horizon geometry of a given black hole should encode information only about the particular solution which produces the particular near horizon geometry. Multi-centered black holes have multiple horizons with multiple near horizon geometry, and hence the contribution to their degeneracies should involve studying string theory in the near horizon geometry of each of these black holes.[1]

In order to make this modified proposal concrete we must independently define microscopic degeneracy of a single centered black hole. Typically microscopic computation involves studying degeneracies of various brane configurations and cannot distinguish whether a given state would correspond to a single centered or a multi-centered configuration in the limit when the state becomes a black hole. However in asymptotically flat four dimensional space-time there is a simple algorithm for calculating the spectrum of single centered black holes in the microscopic theory; we simply need to set the asymptotic values of the moduli to be equal to their attractor values.[2] In that case all multi-centered black hole solutions disappear and the microstate counting only picks up the contribution from the single centered black holes.

In this paper we focus on a different counterexample that cannot be resolved by invoking the existence of multi-centered black holes. This involves the BMPV black hole [35], whose microscopic description involves a D1-D5 system of type IIB string theory on $K3\times S^1$, carrying momentum along S^1 and equal angular momentum in two planes transverse to the D5-brane. The macroscopic description of this is a five dimensional rotating black hole. By placing this black hole at the center of a Taub-NUT space we get a four dimensional black hole [36]. Since near the origin the Taub-NUT space appears as flat space, the near horizon geometries of the four and five dimensional black holes are exactly identical [37, 38]. However the microscopic description of the four dimensional black hole involves D1-D5-brane moving in the background of a Kaluza-Klein monopole and the degeneracies of this system are different from those of just the D1-D5 system [20]. This would seem to contradict the claim that the microscopic degeneracies of single centered black holes are completely encoded in their near horizon geometries.

[1]The near horizon AdS_2 geometry of a black hole can fragment into multiple throats carrying different charges [32–34]. However for such solutions the charges carried by the fragments are mutually local, i.e. have $(q_i \cdot p_j - q_j \cdot p_i) = 0$ where (q_i, p_i) denote the electric and magnetic charge vectors of the ith throat. Since such configurations do not contribute to the entropy [28, 30], the conclusion that the near horizon geometry of a black hole captures the degeneracies of single centered black holes remains unchanged.

[2]This is sufficient but not necessary; all we need is that the asymptotic values of the moduli should be chosen such that we can continuously deform them to the attractor values without crossing any wall of marginal stability.

We suggest the following resolution of this puzzle. Common sense tells us that the near horizon geometry should capture the degeneracies associated with the dynamics of the horizon. If the black hole has no hair, that is no degree of freedom living outside the horizon that could contribute to the degeneracy, then the near horizon geometry would capture the complete information about the microscopic degeneracy of the black hole. However if the black hole solution contains degrees of freedom living outside the horizon then the full degeneracy of the black hole has to be computed by combining the contribution from the horizon with the contribution from the degrees of freedom living outside the horizon, and the combined contribution will then have to be compared with the microscopic degeneracies. Thus two black holes having identical near horizon geometry can have different microscopic degeneracies if they have different sets of degrees of freedom living outside the horizon. We expect that at least for extremal black holes the separation between the contribution from the black hole hair and the contribution from the horizon degrees of freedom can be done rigorously since the horizon is separated from the asymptotic space-time by an infinite throat. Thus two such extremal black holes with identical near horizon geometry will have identical degeneracies of microstates *after we remove the contribution from the degrees of freedom living outside the horizon.*[3]

In the rest of the paper we shall identify the degrees of freedom living outside the horizon for both the BMPV black hole and the four dimensional extremal black hole obtained by placing the BMPV black hole in a Taub-NUT geometry, and then show that their microscopic degeneracies agree after we remove the contribution due to the hair. The organisation of the sections will be as follows. In section 2 we identify the hair degrees of freedom of the five dimensional BMPV black hole, and remove their contribution from the partition function to determine the partition function associated with the horizon degrees of freedom. The result is given in (2.17). In section 3 we repeat the same analysis for the four dimensional black hole obtained by placing the BMPV black hole at the center of Taub-NUT space. The result, given in (3.20), is found to agree with (2.17). It of course remains a challenge to reproduce these microscopic results from a macroscopic calculation, e.g. of the quantum entropy function. In the two appendices we describe explicit construction of the bosonic modes associated with the hair degrees of freedom.

Before concluding this section we would like to add a word of caution. While we have identified appropriate hair degrees of freedom for the five and four dimensional black holes after whose removal the result for the partition function of the two black holes agree, we have not proved that these are the only hair degrees of freedom. If there are additional hair degrees of freedom which differ for these two black holes then it could spoil the agreement. On the other hand if there are additional hair degrees of freedom which are common to both black holes then the agreement between the partition functions of the two black holes after hair removal will continue to hold.

[3] This is similar in spirit to the phenomenon that for a stack of N D3-branes, string theory living in the bulk of the near horizon $AdS_5 \times S^5$ geometry does not capture the U(1) center of mass degrees of freedom of the D3-branes [39].

JHEP07(2009)091

2 Analysis of the BMPV black hole entropy

We begin with the analysis of microscopic degeneracy of the five dimensional quarter BPS black hole in type IIB string theory on $K3$. The microscopic description involves Q_5 number of D5-branes wrapped on $K3 \times S^1$ and Q_1 number of D1-branes wrapped on S^1 carrying $-n$ units of momentum along S^1 (with $n > 0$) and J units of angular momentum. For simplicity we shall take $Q_5 = 1$ without any loss of generality since the result depends on Q_1 and Q_5 only through the combination $Q_5(Q_1 - Q_5)$. Our convention for angular momentum and supersymmetry generators will be as follows. We denote the SO(4) rotation group of the five dimensional space-time by $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and identify the angular momentum J with twice the diagonal generator of $\mathrm{SU}(2)_L$. We also denote by h the eigenvalue of the diagonal generator of $\mathrm{SU}(2)_R$. Since supersymmetry transformation parameters of type IIB on K3 are chiral spinors in six dimensions, when we regard them as representations of the $\mathrm{SO}(1,1) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ subgroup of the Lorentz group, with $\mathrm{SO}(1,1)$ acting on the common direction of the D1-brane and the D5-brane, the $\mathrm{SO}(1,1)$ quantum numbers will be correlated with the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ quantum numbers. We shall now argue that in order that the configuration described above describes a quarter BPS state, we must choose the convention that the left-chiral spinors of $\mathrm{SO}(1,1)$ carry $(J = 0, 2h = \pm 1)$ and the right-chiral spinors of $\mathrm{SO}(1,1)$ carry $(J = \pm 1, h = 0)$. The argument goes as follows. First of all note that since the D1-D5-brane system carries negative momentum along S^1, it must be allowed to carry left-moving excitations without violating supersymmetry. Thus the left-chiral excitations must be neutral under the unbroken supersymmetries of the system. This in turn implies that these supersymmetry transformation parameters must be left-chiral spinors of $\mathrm{SO}(1,1)$, — since left-chiral supersymmetry transformation parameters act on the right-chiral modes and vice versa. We shall now argue that the unbroken supersymmetry transformation parameters must also carry $J = 0$, — this would force us to choose the convention described above. In order that the system can carry macroscopic J charge, a large number of internal modes must carry non-vanishing J charge. Now most of the bosonic degrees of freedom come from the motion of the D1-brane inside the D5-brane, i.e. along the $K3$ direction. This leads to four bosons for each D1-brane describing its position along $K3$. These modes are clearly neutral under the SO(4) rotation along the space transverse to the D1-D5-brane system, and hence do not carry any J charge. On the other hand for every D1-brane we also have eight fermionic modes, – four carrying $(J \neq 0, h = 0)$ and four carrying $(J = 0, h \neq 0)$.[4] The requirement of unbroken supersymmetry freezes the modes on which supersymmetry acts, i.e. those which form partners of the bosons. Now since we want to excite the modes carrying J charge, we must freeze the ones with $J = 0$. Thus the latter must be acted upon by supersymmetry and paired with the bosons. Since the bosons carry $J = 0$, the supersymmetry transformation parameter must also carry $J = 0$. This establishes the desired result.

We denote by $d_{5D}(n, Q_1, J)$ the helicity trace $-Tr\left((-1)^{2h+J}\,(2h)^2\right)/2!$ of five dimen-

[4]These have opposite relation between the $\mathrm{SO}(1,1)$ and $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ quantum numbers, but we shall not need to use this information here.

sional black hole carrying quantum numbers (n, Q_1, J), and define

$$Z_{5D}(\rho,\sigma,v) = \sum_{n,Q_1,J} d_{5D}(n,Q_1,J)\, \exp\left[2\pi i\{(Q_1-1)\,\sigma + (n-1)\,\rho + Jv\}\right] . \tag{2.1}$$

The -1 in $(Q_1 - 1)$ reflects the fact that a D5-brane wrapped on $K3$ carries -1 units of D1-brane charge. On the other hand the -1 in $(n-1)$ has been introduced due to the fact that this charge measured at ∞ differs from that measured on the horizon [40–42] – a Chern-Simons coupling in the action produces -1 unit of this charge from the region between the horizon and infinity. Thus if $-n$ is the total momentum along S^1 carried by the black hole, the charge measured at the horizon will be $-(n-1)$. Explicit computation shows that Z_{5D} defined in (2.1) has the form

$$\begin{aligned} Z_{5D}(\rho,\sigma,v) = {}& e^{-2\pi i\rho-2\pi i\sigma} \prod_{\substack{k,l,j\in\mathbb{Z}\\ k\geq 1, l\geq 0}} \left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)} \\ &\times\left\{\prod_{l\geq 1}(1-e^{2\pi i(l\rho+v)})^{-2}\,(1-e^{2\pi i(l\rho-v)})^{-2}\,(1-e^{2\pi il\rho})^4\right\}(-1)\,(e^{\pi iv}-e^{-\pi iv})^2 \\ &+e^{-2\pi i\rho-2\pi i\sigma} \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1-e^{2\pi i(\sigma k+vj)}\right)^{-c(-j^2)} (e^{\pi iv}-e^{-\pi iv})^2 , \end{aligned} \tag{2.2}$$

where the coefficients $c(n)$ are defined via the equation

$$8\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}+\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}+\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right] = \sum_{j,n\in\mathbb{Z}} c(4n-j^2)\, e^{2\pi in\tau+2\pi ijz} . \tag{2.3}$$

Eq. (2.2) requires some explanation. The first line of (2.2) denotes the contribution from the relative motion of the D1-D5 system and was computed in [43]. The asymptotic expansion of the degeneracies of this system has been studied recently in [41, 44]. The second line represents contribution from the 'center of mass modes' of the D1-D5 system. This contribution can be calculated as follows. Since the D1-D5 system breaks the translation symmetries along the four directions transverse to the brane, the (1+1) dimensional world-volume theory of this system, spanned by the time coordinate and the coordinate along S^1, will contain four goldstone bosons associated with the four broken translation generators. Furthermore since the ground state of the D1-D5 system also breaks eight out of the sixteen supersymmetries of type IIB string theory on K3, we shall have eight goldstino fermions carrying the same quantum numbers as the broken supersymmetry transformation parameters. This leads to four left-moving and four right-moving fermions living on the D1-D5-brane world-volume. In our convention the left-moving fermions carry $(J = 0, 2h = \pm 1)$ and the right-moving fermions carry $(J = \pm 1, 2h = 0)$. We need to count excitations of this system preserving four supersymmetries, parametrized by left-chiral spinors on the D1-D5-brane world-volume. Since these transformations act on the right-moving fermions and bosons, the BPS condition will freeze all the right-moving excitations except the zero modes. Since the right-moving fermions carry $J = \pm 1$, $h = 0$, quantization of a pair of right chiral zero modes would produce a pair of states with $J = \pm\frac{1}{2}$, $h = 0$. Thus the net contribution of four right chiral zero modes to the trace, containing a factor of $(-1)^J e^{2\pi i v J} = e^{2\pi i J(v+\frac{1}{2})}$, is a factor of $(e^{\pi i(v+\frac{1}{2})} + e^{-\pi i(v+\frac{1}{2})})^2 = -(e^{\pi i v} - e^{-\pi i v})^2$. This accounts for the last two factors in the second line of (2.2). The BPS condition does not restrict the left-moving degrees of freedom and the terms in the curly bracket in the second line of (2.2) represent contribution from these left-moving excitations. In particular the zero modes of the left-moving fermions, carrying helicities $\pm 1/2$, soak up the factors of $-(2\,h)^2/2!$ in the helicity trace so that if we leave aside these zero modes, contribution to the helicity trace from the rest of the modes involve computing the Witten index $Tr(-1)^F$. Since the left-moving fermions have $J = 0$, their oscillators lead to the last term in the product inside the curly bracket. On the other hand the left-moving bosons, transforming under $(2,2)$ representation of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, carry ± 1 units of J quantum numbers and lead to the first two terms inside the curly bracket. Finally the term in the last line of (2.2) removes the contribution of the $n = 0$ term[5] from eq. (2.1), i.e. it subtracts the term whose ρ dependence is of the form $e^{-2\pi i \rho}$. The rationale for subtracting this term is that for $n = 0$ the D1-D5 system includes contribution from half-BPS states. Thus it is more natural to consider the partition function of pure quarter BPS states by subtracting the contribution due to the $n = 0$ term.

Now we need to analyze the contribution to the partition function from the degrees of freedom of the black hole living outside the horizon and remove this contribution from (2.2) to determine the expected microscopic degeneracies associated with the horizon. We begin by writing down the action and the black hole solution. The relevant part of the action containing the string metric $G_{\mu\nu}$, dilaton Φ and the Ramond-Ramond 3-form field strength $F^{(3)} = dC^{(2)}$ takes the form

$$\frac{1}{(2\pi)^7} \int d^{10}x \sqrt{-\det G} \left[e^{-2\Phi} \left(R + 4\, G^{\mu\nu}\, \partial_\mu \Phi \partial_\nu \Phi \right) - \frac{1}{12} F^{(3)}_{MNP} F^{(3)MNP} \right], \tag{2.4}$$

in $\alpha' = 1$ unit. For simplicity we shall set the asymptotic values of the moduli to their attractor values for the specific black hole solution we analyze, so that all the moduli fields including the dilaton are constants. The generalization to more general asymptotic values is straightforward. In this case the rotating black hole solution describing Q_5 D5-branes along $K3 \times S^1$, Q_1 D1-branes along S^1, $-n$ units of momentum along S^1 and angular momentum J, takes the form[6]

$$\begin{aligned}
dS^2 &= \left(1 + \frac{r_0}{r}\right)^{-1} \left[-dt^2 + (dx^5)^2 + \frac{r_0}{r}(dt + dx^5)^2 + \frac{\widetilde{J}}{4r}\, (dt + dx^5)\, (dx^4 + \cos\theta\, d\phi) \right] \\
&\quad + \widehat{g}_{mn}(u)\, du^m du^n + \left(1 + \frac{r_0}{r}\right) ds^2_{\text{flat}}\,, \\
ds^2_{\text{flat}} &= r\, (dx^4 + \cos\theta d\phi)^2 + \frac{1}{r}\, (dr^2 + r^2 d\theta^2 + r^2\, \sin^2\theta\, d\phi^2)\,,
\end{aligned}$$

[5]Throughout this paper we shall denote the additive term proportional to $e^{-2\pi i \rho}$ as the $n = 0$ term.

[6]Conventionally the BMPV black hole as well as the BMPV black hole at the center of Taub-NUT space is expressed as a solution in five dimensional supergravity theory [35–37]. Here we express them as solutions in a ten dimensional theory so that we can study the excitations which propagate along the internal directions.

JHEP07(2009)091

$$
\begin{aligned}
(\theta,\phi,x^4) &\equiv (2\pi-\theta,\phi+\pi,x^4+\pi) \equiv (\theta,\phi+2\pi,x^4+2\pi) \equiv (\theta,\phi,x^4+4\pi)\,,\\
e^{\Phi} &= \lambda\,,\\
F^{(3)} &\equiv \frac{1}{6}F^{(3)}_{MNP}dx^M\wedge dx^N\wedge dx^P\\
&= \frac{r_0}{\lambda}\left(\epsilon_3+*_6\epsilon_3+\frac{1}{r_0}\left(1+\frac{r_0}{r}\right)^{-1}(dx^5+dt)\wedge d\zeta\right)\,,\\
\epsilon_3 &\equiv \sin\theta\, dx^4\wedge d\theta\wedge d\phi\,,
\end{aligned}
\tag{2.5}
$$

where x^5 is the coordinate of the circle S^1 with period $2\,\pi\,R_5$, u^m for $m=6,\ldots,9$ are the coordinates of $K3$, $\widehat{g}_{mn}$ is the metric on $K3$, $(2\pi)^4\,V$ is the volume of $K3$ measured in this metric, λ is the asymptotic value of the string coupling, $*_6$ denotes Hodge dual in the six dimensions spanned by t, x^5, x^4, r, θ and ϕ with the convention $\epsilon^{t54r\theta\phi}=1$, and

$$
r_0 = \frac{\lambda(Q_1-Q_5)}{4V} = \frac{\lambda Q_5}{4} = \frac{\lambda^2|n|}{4R_5^2V}\,, \tag{2.6}
$$

$$
\widetilde{J} = \frac{J\,\lambda^2}{2\,R_5\,V}\,, \tag{2.7}
$$

$$
\zeta = -\frac{\widetilde{J}}{8r}\,(dx^4+\cos\theta d\phi)\,. \tag{2.8}
$$

Eq. (2.6) determines the asymptotic moduli V and λ/R_5^2 in terms of the charges. This corresponds to setting the asymptotic moduli to their attractor values. $ds^2_{\rm flat}$ describes flat euclidean space in the Gibbons-Hawking coordinates. Higher derivative corrections to the entropy of this black hole have been discussed extensively in [40–42, 45–49].

Now the black hole solution breaks four translation symmetries and twelve of the sixteen space-time supersymmetries, and hence we expect to have four bosonic zero modes and twelve fermionic zero modes living on the black hole, forming part of the black hole hair.[7] Typically the quantization of the bosonic zero modes do not give rise to additional degeneracies but produces new charge sectors instead, — this was illustrated in [50] in the context of four dimensional black holes. However the quantization of the fermion zero modes does affect the partition function. The (J,h) quantum numbers of the fermion zero modes can be read out by comparison with the microscopic description. Since the four unbroken supersymmetries are labelled by left-chiral spinors on the D1-D5 world-volume, eight of the broken supersymmetries are right-chiral and four of the broken supersymmetries are left-chiral. This leads to eight right-chiral and four left-chiral zero modes. The left-chiral zero modes carrying $(J=0,h=\pm\frac{1}{2})$ soak up the factors of $-(2\,h)^2/2!$ in the helicity trace, so that for the rest of the degrees of freedom we only need to calculate the Witten index $Tr(-1)^{2h+J}$. On the other hand the right-chiral zero modes carry $(J=\pm 1,h=0)$ and their contribution to the partition function is given by

$$
(e^{\pi i v}-e^{-\pi i v})^4\,. \tag{2.9}
$$

[7] Given that black hole solution outside the horizon changes under these translations and supersymmetry transformations, it is clear that these modes are non-vanishing outside the horizon. What is not apparent at this stage is whether they have support entirely outside the horizon. For now we shall proceed by assuming that this is the case, but will study this issue in detail in appendix A.

This however is not the end of the story. Given a zero mode we can explore whether it is possible to lift it to a full fledged field in $(1+1)$ dimensions spanned by the coordinates (t, x^5). If we can lift them to such fields then the oscillation modes of these fields would produce additional contribution to the partition function of the black hole hair. To this end we note that if the black hole solution had been Lorentz invariant in the (x^5, t) plane, then any broken symmetry would automatically lead to a massless goldstone or goldstino field on the black hole world volume instead of just the zero modes. In particular the bosonic zero modes would lift to scalar fields, left-chiral fermion zero modes would lift to left-moving fermion fields and right-chiral fermion zero modes would lift to right-moving fermion fields. However the black hole solution (2.5) does not have $(1+1)$ dimensional Lorentz invariance, and hence a priori we cannot use results in 1+1 dimensonal quantum field theory to conclude that associated with a broken symmetry we shall have a massless field living on the world-volume of the black hole. Nevertheless we shall now argue that the left-moving modes are not affected by the breaking of Lorentz invariance and continue to exist. Our argument will be somewhat heuristic, but we compensate for it by giving a detailed construction of these modes in appendix A. First we note that the source of Lorentz non-invariance in (2.5) are the $(dt + dx^5)^2$ term and the $(dt + dx^5)\,(dx^4 + \cos\theta\, d\phi)$ terms in the metric. This structure of the metric shows that only the g_{++} and g_{+i} components of the metric violate the Lorentz invariance. Since these lead to g^{--} and g^{-i} components of the metric but no g^{++} or g^{+i} components, we see that the Lorentz violating terms in the equation of motion of various modes around the solution must involve ∂_- derivatives or $_{-\dots}$ components of fields. In particular the left-moving fields φ for which $\partial_-\varphi = 0$ do not couple to the g^{--} or g^{-i} components of the metric and should continue to describe solutions to linearized equations of motion around the black hole background. Thus we can conclude that the world-volume of the black hole will have four left-moving bosonic fields carrying $(J = \pm 1, 2h = \pm 1)$ and four left-moving fermion fields carrying $(J = 0, 2h = \pm 1)$. Their contribution to the partition function is given by

$$\prod_{l\geq 1}(1 - e^{2\pi i(l\rho+v)})^{-2}\,(1 - e^{2\pi i(l\rho-v)})^{-2}\,(1 - e^{2\pi i l\rho})^4\,. \tag{2.10}$$

Multiplying this by the contribution (2.9) from the zero modes we get the total contribution to the partition function from the degrees of freedom living outside the horizon

$$Z^{\rm hair}_{5D}(\rho,\sigma,v) = (e^{\pi i v} - e^{-\pi i v})^4 \prod_{l\geq 1}(1 - e^{2\pi i(l\rho+v)})^{-2}\,(1 - e^{2\pi i(l\rho-v)})^{-2}\,(1 - e^{2\pi i l\rho})^4\,. \tag{2.11}$$

Let $Z^{\rm hor}_{5D}(\rho,\sigma,v)$ denote the partition function associated with the horizon degrees of freedom of the five dimensional black hole. Naively we have the relation $Z_{5D} = Z^{\rm hor}_{5D} \times Z^{\rm hair}_{5D}$. However we shall now argue that there is an extra additive contribution to Z_{5D}, and the correct relation is

$$Z_{5D} = Z^{\rm hor}_{5D} \times Z^{\rm hair}_{5D} + Z^{\rm extra}_{5D}\,. \tag{2.12}$$

The extra contribution $Z^{\rm extra}_{5D}$ comes from starting with a configuration where the black hole does not carry any momentum along S^1, and then exciting its hair degrees of freedom

JHEP07(2009)091

carrying momentum. As can be seen from (2.6), the initial configuration is singular in the supergravity approximation. Thus it describes a 'small black hole' in five dimensions,[8] and hence its hair degrees of freedom are different from the ones we analyzed earlier. In particular since the D1-D5 system without momentum breaks only four left-chiral and four right chiral supersymmetries, we have only four right chiral zero modes instead of 8, and hence a factor of $-(e^{\pi iv} - e^{-\pi iv})^2$ will be missing from the hair degrees of freedom. Furthermore since the D1-D5-brane world-volume theory now has full (1+1) dimensional Lorentz invariance, the right-chiral modes are now lifted to full right-moving fields, However the requirement of unbroken supersymmetry still freezes the right-moving excitations to their ground state. Thus the net contribution from the hair is given by

$$Z^{\rm hair}_{\rm small} = -(e^{\pi iv} - e^{-\pi iv})^2 \prod_{l\geq 1} (1 - e^{2\pi i(l\rho+v)})^{-2} \, (1 - e^{2\pi i(l\rho-v)})^{-2} \, (1 - e^{2\pi il\rho})^4 \, . \tag{2.13}$$

Let us denote by $Z^{\rm hor}_{\rm small}$ the contribution from the horizon degrees of freedom of the small black hole. Then $Z^{\rm extra}_{5D}$ will be obtained by taking the product $Z^{\rm hor}_{\rm small} \times Z^{\rm hair}_{\rm small}$ and subtracting the $n = 0$ contribution. On the other hand $Z^{\rm hor}_{\rm small}$ may be determined by identifying the $n = 0$ contribution in $Z^{\rm hor}_{\rm small} \times Z^{\rm hair}_{\rm small}$ with the partition function of the D1-D5 system with no momentum along S^1. The latter is simply the negative of the last term in (2.2):

$$-\, e^{-2\pi i\rho - 2\pi i\sigma} \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1 - e^{2\pi i(\sigma k + vj)}\right)^{-c(-j^2)} \, (e^{\pi iv} - e^{-\pi iv})^2 \, . \tag{2.14}$$

Dividing (2.14) by the ρ independent term in the series expansion of (2.13) gives

$$Z^{\rm hor}_{\rm small}(\rho,\sigma,v) = e^{-2\pi i\rho - 2\pi i\sigma} \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1 - e^{2\pi i(\sigma k + vj)}\right)^{-c(-j^2)} \, . \tag{2.15}$$

$Z^{\rm extra}_{5D}$ is now obtained by multiplying (2.15) by (2.13) and then subtracting the $n = 0$ term, i.e. the term proportional to $e^{-2\pi i\rho}$ in the series expansion:

$$\begin{aligned} Z^{\rm extra}_{5D}(\rho,\sigma,v) = & -e^{-2\pi i\rho - 2\pi i\sigma} \, (e^{\pi iv} - e^{-\pi iv})^2 \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1 - e^{2\pi i(\sigma k + vj)}\right)^{-c(-j^2)} \\ & \times \prod_{l\geq 1} (1 - e^{2\pi i(l\rho+v)})^{-2} \, (1 - e^{2\pi i(l\rho-v)})^{-2} \, (1 - e^{2\pi il\rho})^4 \\ & + e^{-2\pi i\rho - 2\pi i\sigma} \, (e^{\pi iv} - e^{-\pi iv})^2 \prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}} \left(1 - e^{2\pi i(\sigma k + vj)}\right)^{-c(-j^2)} \, . \end{aligned} \tag{2.16}$$

[8] Here, as well as in section 3, we shall denote by 'small black hole' any object which is singular in the supergravity limit, carrying Q_1, Q_5 and J quantum numbers but no momentum along S^1. Thus it includes small black ring configurations as well [51, 52].

Using (2.2), (2.11), (2.12) and (2.16) we now get

$$
\begin{aligned}
Z_{5D}^{\rm hor}(\rho,\sigma,v) &= (Z_{5D} - Z_{5D}^{\rm extra})/Z_{5D}^{\rm hair} \\
&= -e^{-2\pi i\rho - 2\pi i\sigma}\left(e^{\pi i v} - e^{-\pi i v}\right)^{-2} \prod_{\substack{k,l,j\in\mathbb{Z}\\ k\ge 1, l\ge 0}} \left(1 - e^{2\pi i(\sigma k + \rho l + v j)}\right)^{-c(4lk - j^2)} \\
&\quad + e^{-2\pi i\rho - 2\pi i\sigma}\left(e^{\pi i v} - e^{-\pi i v}\right)^{-2} \prod_{\substack{k,j\in\mathbb{Z}\\ k\ge 1}} \left(1 - e^{2\pi i(\sigma k + v j)}\right)^{-c(-j^2)} .
\end{aligned} \tag{2.17}
$$

The presence of the $(e^{\pi i v} - e^{-\pi i v})^{-2}$ factor may lead one to believe that $Z_{5D}^{\rm hor}$ has a double pole at $v = 0$ and hence the index extracted from this partition function will suffer from the contour prescription ambiguities discussed in [22–24]. However using the relation $\sum_j c(4n - j^2) = 24\,\delta_{n,0}$ and the $v \to -v$ symmetry one can show that the sum of the two terms in (2.17) has no singularity at $v = 0$. Thus (2.17) leads to an unambiguous result for the index of quarter BPS states associated with the horizon degrees of freedom. We also note that since the factor of $-(2\,h)^2/2!$ in the helicity trace is soaked up by the fermion zero modes associated with the hair, the partition function $Z_{5D}^{\rm hor}$ measures the Witten index $Tr(-1)^F = Tr(-1)^{2h+J}$ of the black hole microstates associated with the horizon in a given (n, Q_1, J) sector.

3 Analysis of the four dimensional black hole entropy

Now we turn to the degeneracies of four dimensional black holes obtained by placing the five dimensional black hole described above at the center of Taub-NUT space. The corresponding solution is given by [36]

$$
\begin{aligned}
dS^2 &= \left(1 + \frac{r_0}{r}\right)^{-1}\Bigg[- dt^2 + (dx^5)^2 + \frac{r_0}{r}(dt + dx^5)^2 \\
&\qquad + \frac{\widetilde{J}}{4}\left(\frac{1}{r} + \frac{4}{R_4^2}\right)(dx^4 + \cos\theta\, d\phi)\,(dt + dx^5)\Bigg] \\
&\quad + \widehat{g}_{mn}\, du^m du^n + \left(1 + \frac{r_0}{r}\right) ds_{TN}^2 , \\
e^{\Phi} &= \lambda , \\
F^{(3)} &= \frac{r_0}{\lambda}\left(\epsilon_3 + *_6\epsilon_3 + \frac{1}{r_0}\left(1 + \frac{r_0}{r}\right)^{-1}(dx^5 + dt)\wedge d\widetilde{\zeta}\right) ,
\end{aligned} \tag{3.1}
$$

where

$$
\widetilde{\zeta} = -\frac{\widetilde{J}}{8}\left(\frac{1}{r} + \frac{4}{R_4^2}\right)(dx^4 + \cos\theta d\phi) , \tag{3.2}
$$

$$
ds_{TN}^2 = \left(\frac{4}{R_4^2} + \frac{1}{r}\right)^{-1}(dx^4 + \cos\theta d\phi)^2 + \left(\frac{4}{R_4^2} + \frac{1}{r}\right)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) . \tag{3.3}
$$

Here R_4 is a constant labelling the asymptotic radius of the x^4 circle. Note that for $R_4^2 = 4r_0$ the 44, 45 and 55 components of the metric become constant independent of r.

Thus $4r_0$ is the attractor value of R_4^2. We shall proceed with the solution for general R_4. Using (3.3) we can express the solution given in (3.1) as

$$\begin{aligned} dS^2 &= -e_0^2+e_1^2+e_2^2+e_3^2+e_4^2+e_5^2+\widehat{g}_{mn}\,du^m\,du^n\,, \\ F^{(3)} &= \frac{r_0}{\lambda\, r^2}\left[\left(1+\frac{r_0}{r}\right)^{-3/2}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)^{-1/2}(e_2\wedge e_4\wedge e_5+e_0\wedge e_1\wedge e_3)\right. \\ &\quad \left. +\frac{\widetilde{J}}{8\,r_0}\left(1+\frac{r_0}{r}\right)^{-2}(-e_0\wedge e_2\wedge e_3+e_0\wedge e_4\wedge e_5-e_1\wedge e_2\wedge e_3+e_1\wedge e_4\wedge e_5)\right]\,, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} e_0 &= \left(1+\frac{r_0}{r}\right)^{-1}(dt+\widetilde{\zeta}), \\ e_1 &= \left(dx^5+dt-\left(1+\frac{r_0}{r}\right)^{-1}(dt+\widetilde{\zeta})\right), \\ e_2 &= \left(1+\frac{r_0}{r}\right)^{1/2}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)^{-1/2}(dx^4+\cos\theta d\phi), \\ e_3 &= \left(1+\frac{r_0}{r}\right)^{1/2}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)^{1/2} dr\,, \\ e_4 &= \left(1+\frac{r_0}{r}\right)^{1/2}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)^{1/2} r\,d\theta\,, \\ e_5 &= \left(1+\frac{r_0}{r}\right)^{1/2}\left(\frac{1}{r}+\frac{4}{R_4^2}\right)^{1/2} r\,\sin\theta\,d\phi\,. \end{aligned} \tag{3.5}$$

Since x^4 has period 4π, the asymptotic circle parametrized by x^4 has finite radius. Thus asymptotically we have four non-compact space-time dimensions. Also since x^4 now represents a compact coordinate, the quantum number J is interpreted as the momentum along x^4 instead of angular momentum. However for small r the solution approaches that given in (2.5), and both solutions have identical near horizon geometry. To see this explicitly we take the near horizon limit by first defining new coordinates (ρ,τ,y) via

$$r=r_0\,\beta\rho, \quad t=\tau/\beta, \quad x^5=y-t \tag{3.6}$$

and taking the limit $\beta\to 0$. In this limit both (2.5) and (3.1) take the form[9]

$$\begin{aligned} dS^2 &= r_0\frac{d\rho^2}{\rho^2}+dy^2+r_0(dx^4+\cos\theta d\phi)^2+\frac{\widetilde{J}}{4r_0}dy(dx^4+\cos\theta d\phi)-2\rho dy d\tau \\ &\quad +r_0\left(d\theta^2+\sin^2\theta d\phi^2\right)+\widehat{g}_{mn}du^m du^n\,, \\ e^{\Phi} &= \lambda\,, \\ F^{(3)} &= \frac{r_0}{\lambda}\left[\epsilon_3+*\epsilon_3+\frac{\widetilde{J}}{8\,r_0^2}\,dy\wedge\left(\frac{1}{\rho}\,d\rho\wedge(d\,x^4+\cos\theta\,d\phi)+\sin\theta\,d\theta\wedge d\phi\right)\right]\,. \end{aligned} \tag{3.7}$$

[9]We could take a more careful limit by beginning with a non-extremal black hole and scaling the non-extremality parameter also by β as reviewed in [14]. However this does not play any role in the present discussion.

Thus we expect that the contribution to the degeneracy from the horizon degrees of freedom will be identical for the four and the five dimensional black holes. In particular the quantum entropy function will give identical results for the two solutions. We shall now try to test this at the microscopic level by computing the degeneracies associated with the four dimensional black hole horizon.

The microscopic degeneracy associated with the four dimensional black hole is different from that of the five dimensional black hole, as it receives additional contribution from the modes living on the Taub-NUT space as well as the modes associated with the motion of the D1-D5-brane in the Taub-NUT space [20]. If we denote by $d_{4D}(n, Q_1, J)$ the sixth helicity trace[10] $-B_6 \equiv -Tr((-1)^{2h+J}(2h)^6)/6!$ for the states of the four dimensional black hole carrying quantum numbers (n, Q_1, J) then the four dimensional partition function defined via

$$Z_{4D}(\rho, \sigma, v) = \sum_{n,Q_1,J} d_{4D}(n-1, Q_1, J) \, \exp\left[2\pi i\{(Q_1 - 1)\,\sigma + (n-1)\,\rho + Jv\}\right] , \qquad (3.8)$$

has the form [20, 38, 53–55][11]

$$Z_{4D}(\rho, \sigma, v) = -e^{-2\pi i\rho - 2\pi i\sigma - 2\pi i v} \prod_{\substack{k,l,j\in\mathbb{Z} \\ k,l\geq 0, j<0\, for\, k=l=0}} \left(1 - e^{2\pi i(\sigma k + \rho l + vj)}\right)^{-c(4lk - j^2)} . \qquad (3.9)$$

Note that we now have $(n-1)$ in the argument of d_{4D} in (3.8), matching the coefficient of ρ in the exponent. This reflects the fact that for the four dimensional black holes the charge measured at the horizon agrees with the charge measured by an asymptotic observer. The $e^{-2\pi i\rho}$ factor in (3.9) is a reflection of the fact that the ground state of the Taub-NUT space carries -1 unit of momentum along S^1; however this is visible only after taking into account the higher derivative term in the action involving the gravitational Chern-Simons term. Finally we note that there is no need to subtract the $n = 0$ contribution from the sum, since in the presence of a Taub-NUT space even the $n = 0$ states are quarter BPS. The near horizon geometry of the $n = 0$ black hole will however lose the memory of the Taub-NUT background and will have enhanced supersymmetries.

We now need to remove the contribution to Z_{4D} from the degrees of freedom living outside the horizon. We begin by counting the fermionic modes living outside the horizon. First of all, there are 12 broken supersymmetry generators leading to 12 fermion zero modes. They carry $h = \pm\frac{1}{2}$ and soak up the $-(2\,h)^6/6!$ factor from the helicity trace. Thus the effect of removing their contribution is to map the helicity trace index to the Witten index of the remaining system [20, 56]. Had the black hole world-volume theory been Lorentz invariant in the (x^5, t) coordinates, eight of the zero modes would lift to right-moving fermion fields and four of the zero modes would lift to left-moving fermion fields on the black hole world-volume. As in the case of five dimensional black holes, we expect

[10] h now denotes the third component of the angular momentum in the (3+1) dimensional theory. J represents a U(1) charge in the four dimensional theory and its inclusion in the trace is purely a matter of convenience.

[11] The correct sign of the partition function has been determined in [56]. Note that $d_{4D}(n, Q_1, J)$ used here differ from the index used in [56] by a factor of $(-1)^J$ due to the insertion of $(-1)^J$ in our definition of B_6. However the definition of partition function in [56] has an explicit factor of $(-1)^{J+1}$ inserted.

that the breaking of Lorentz invariance does not affect the equations for the left-moving modes and hence we should be able to lift the four left-chiral fermion zero modes into full fledged left-moving fermion fields on the black hole world-volume. These modes produce a contribution to the Witten index of the form

$$\prod_{l=1}^{\infty}(1-e^{2\pi i l\rho})^4 \,. \tag{3.10}$$

Next we turn to the bosonic modes living on the black hole. As before we shall proceed by pretending that the black hole world-volume has Lorentz invariance in the (x^5,t) plane, and then take into account the lack of Lorentz invariance by freezing the right-moving fields. Our arguments will be heuristic, but we give more explicit construction of some of the modes in appendix B. The black hole solution given in (3.1) admits a normalizable closed 2-form inherited from the normalizable harmonic 2-form of the Taub-NUT space [57, 58]. It is given by

$$\omega = -\frac{r}{4r+R_4^2}\sin\theta d\theta\wedge d\phi + \frac{R_4^2}{(4r+R_4^2)^2}\,dr\wedge(dx^4+\cos\theta d\phi)\,. \tag{3.11}$$

Using the metric (3.1) one can easily check that this harmonic form is supported outside the near horizon throat geometry. Thus any 2-form field along this harmonic form will give rise to a scalar mode living outside the horizon. From the NSNS and RR 2-form fields of type IIB string theory we get two scalar modes. Furthermore the 4-form field with self-dual field strength, reduced on the 22 internal cycles of K3, generate 3 right chiral and 19 left chiral 2-form fields in type IIB string theory on K3.[12] Picking up the components of these fields along the 2-form ω we get 19 left-moving scalars and 3 right-moving scalars on the black hole world-volume. By the logic given earlier we expect the left-moving modes to survive even after taking into account the breaking of the Lorentz invariance in the (x^5-t) plane. Besides these there are three goldstone bosons associated with the three broken translational symmetries. After freezing the right-moving modes we get three more left-moving modes on the black hole world-volume. Thus we have altogether 2+19+3=24 left-moving scalars living outside the horizon.[13] Since they do not carry any J quantum number (which now corresponds to momentum along x^4), their contribution to the black hole partition function is given by

$$\prod_{l=1}^{\infty}(1-e^{2\pi i l\rho})^{-24} \,. \tag{3.12}$$

We shall now argue that the four dimensonal solution carries four more left-moving bosonic excitations living outside the horizon and carrying J-charge ± 1. Explicit construction of these modes have been discussed in appendix B. Physically these modes represent the motion of the D1-D5 system relative to the Taub-NUT space. Normally if in a composite system we try to displace one component relative to the other there will be a drastic

[12] In our convention the right-chiral 2-form fields have self-dual 3-form field strength and the left-chiral 2-form fields have anti-self-dual 3-form field strength in six dimensions.

[13] Explicit form of these deformations can be found in appendix B.

change in the near horizon geometry and we would not expect such deformations to be described by modes living outside the horizon. However since the Taub-NUT space is non-singular everywhere, the near horizon geometry of a D1-D5-Taub-NUT system is described by that of the D1-D5 system, and hence moving the Taub-NUT space relative to the D1-D5 system should not alter the near horizon geometry. Thus such deformations should be described by modes living outside the horizon. Furthermore since the coordinates labelling the transverse position of the D1-D5 system transform in the vector representation of SO(4), these modes should carry $J = \pm 1$. By the standard argument based on the lack of Lorentz invariance in the $x^5 - t$ plane, we expect the right-moving modes to be frozen but the left-moving modes should be freely excitable. The contribution from these modes to the partition function is given by

$$\prod_{l=1}^{\infty} \left[\left(1 - e^{2\pi i (l\rho + v)}\right)^{-2} \left(1 - e^{2\pi i (l\rho - v)}\right)^{-2} \right] . \tag{3.13}$$

Can there be additional zero modes associated with the motion of the D1-D5-system relative to the Taub-NUT space? The five dimensional black hole world volume in flat transverse space has four left-chiral fermion zero modes with $(J, 2h) = (0, \pm 1)$ and eight right-chiral fermion zero modes with $(J, 2h) = (\pm 1, 0)$, — all living outside the horizon. By an argument similar to the one in the previous paragraph, we expect them to be approximate zero modes even when we place the five dimensional black hole in the Taub-NUT background. The four left-chiral fermion zero modes form part of the 12 goldstino zero modes of the combined system and have already been counted before. Four of the eight right chiral fermion zero modes must form superpartners of the bosonic zero modes describing the motion of the D1-D5-brane system in transverse space. This gives rise to a factor of $-e^{-2\pi i v}(1 - e^{-2\pi i v})^{-2}$ from summing over bound states in the supersymmetric quantum mechanics describing the zero mode dynamics [20, 56, 58, 59]. The other four right-chiral fermion zero modes which are not paired with the bosons under supersymmetry would give a factor of $-(e^{\pi i v} - e^{-\pi i v})^2$ since they carry $J = \pm 1$. Thus these two factors cancel exactly and we do not get any additional contribution to the hair from these zero modes.

Combining (3.10), (3.12) and (3.13) we get the net contribution to the four dimensional black hole partition function from the hair:

$$Z^{\text{hair}}_{4D}(\rho, \sigma, v) = \prod_{l=1}^{\infty} \left[\left(1 - e^{2\pi i l \rho}\right)^{-20} \left(1 - e^{2\pi i (l\rho + v)}\right)^{-2} \left(1 - e^{2\pi i (l\rho - v)}\right)^{-2} \right] . \tag{3.14}$$

Let Z^{hor}_{4D} denote the partition function of the horizon degrees of freedom of the four dimensional black hole. Then naively we have the relation $Z_{4D} = Z^{\text{hor}}_{4D} \times Z^{\text{hair}}_{4D}$, but as in the case of five dimensional black holes, Z_{4D} receives an extra contribution from the configuration where a small five dimensional black hole carrying no momentum along S^1 is placed at the center of the Taub-NUT space and the momentum along S^1 is carried by the hair degrees of freedom. Denoting the extra contribution by Z^{extra}_{4D} we have

$$Z_{4D} = Z^{\text{hor}}_{4D} \times Z^{\text{hair}}_{4D} + Z^{\text{extra}}_{4D} . \tag{3.15}$$

$Z_{4D}^{\rm extra}$ is given by the product of horizon partition function of the small black hole as given in (2.15) and the contribution from the hair degrees of freedom. The latter now consists of four bosons and four left- and four right-moving fermions associated with the motion of the small black hole in Taub-NUT space, and eight right-moving fermions, eight right-movimg bosons and twenty four left-moving bosons associated with the fluctuations in Taub-NUT space. Instead of going through a detailed analysis of these modes we simply note that the number and dynamics of these modes is identical to those describing the dynamics of the Taub-NUT space and the overall motion of the D1-D5 system in Taub-NUT space as discussed in [20, 56]. Thus the partition function associated with the hair degrees of freedom can be read out from [20, 56]. In particular the contribution from the degrees of freedom associated with the overall motion of the D1-D5 system can be read out from eq. (5.2.22) of [56] for $N = 1$:[14]

$$-e^{-2\pi i v}\,(1-e^{-2\pi i v})^{-2}\prod_{l\geq 1}(1-e^{2\pi i(l\rho+v)})^{-2}\,(1-e^{2\pi i(l\rho-v)})^{-2}\,(1-e^{2\pi i l\rho})^{4}\,. \qquad (3.16)$$

On the other hand the degrees of freedom of the Taub-NUT space contributes

$$\prod_{l\geq 1}(1-e^{2\pi i l\rho})^{-24}\,. \qquad (3.17)$$

Taking the product of (2.15), (3.16) and (3.17) gives

$$\begin{aligned} Z_{4D}^{\rm extra}(\rho,\sigma,v) \;=\; & -e^{-2\pi i(v+\rho+\sigma)}\,\left(1-e^{-2\pi i v}\right)^{-2}\prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}}\left(1-e^{2\pi i(\sigma k+vj)}\right)^{-c(-j^2)} \\ & \times\prod_{l=1}^{\infty}\left[\left(1-e^{2\pi i l\rho}\right)^{-20}\left(1-e^{2\pi i(l\rho+v)}\right)^{-2}\left(1-e^{2\pi i(l\rho-v)}\right)^{-2}\right]. \end{aligned} \qquad (3.18)$$

Using (3.9), (3.14), (3.15) and (3.18), and the relations

$$c(0)=20, \qquad c(-1)=2, \qquad c(u)=0 \quad \text{for } u\leq -2\,, \qquad (3.19)$$

we get

$$\begin{aligned} Z_{4D}^{\rm hor}(\rho,\sigma,v) \;=\; & (Z_{4D}-Z_{\rm extra})/Z_{4D}^{\rm hair} \\ =\; & -e^{-2\pi i\rho-2\pi i\sigma}\,(e^{\pi i v}-e^{-\pi i v})^{-2}\prod_{\substack{k,l,j\in\mathbb{Z}\\ k\geq 1,l\geq 0}}\left(1-e^{2\pi i(\sigma k+\rho l+vj)}\right)^{-c(4lk-j^2)} \\ & +e^{-2\pi i\rho-2\pi i\sigma}\,(e^{\pi i v}-e^{-\pi i v})^{-2}\prod_{\substack{k,j\in\mathbb{Z}\\ k\geq 1}}\left(1-e^{2\pi i(\sigma k+vj)}\right)^{-c(-j^2)}\,. \end{aligned} \qquad (3.20)$$

[14] The factor of $-e^{-2\pi i v}\,(1-e^{-2\pi i v})^{-2}$ arises from the sum over bound states of the quantum mechanics describing the motion of the D1-D5-system in Taub-NUT space. The main difference from the computation of $Z_{4D}^{\rm hair}$ is that when the core of the black hole describing the D1-D5 system carries zero momentum, we have only eight fermion zero modes living on the D1-D5 system instead of twelve. Thus an extra factor of $-(e^{\pi i v}-e^{-\pi i v})^2$ is missing here.

This is identical to $Z_{5D}^{\rm hor}$ given in (2.17). We also note that since the $-(2\,h)^6/6!$ term in the trace has been absorbed by the fermion zero modes living outside the horizon, $Z_{4D}^{\rm hor}$ measures the Witten index $Tr(-1)^F$ of the microstates associated with the horizon in a given (n, Q_1, J) sector. The equality of $Z_{4D}^{\rm hor}$ and $Z_{5D}^{\rm hor}$ now shows that the Witten indices associated with the near horizon degrees of freedom of the four and the five dimensional black holes are exactly identical.

Note added. It has been shown in [60] that the hair modes describing the transverse oscillations of the five dimensional black hole, and the oscillations of the BMPV black hole relative to the Taub-NUT space for the four dimensional black hole, develop curvature singularities at the future horizon. Thus they should not be included among the hair degrees of freedom. Since they contributed the same amount to the respective partition functions, the agreement between the partition functions of four and five dimensional black holes after hair removal continue to hold.

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A Explicit construction of the left-moving bosonic modes on the BMPV black hole

Since our argument leading to the existence of left-moving modes on the BMPV black hole has been somewhat abstract we shall now explicitly demonstrate the existence of such modes. For simplicity we shall focus on the left-moving bosonic zero modes associated with the transverse oscillations. If we introduce new coordinates

$$\begin{aligned} w^1 &= 2\sqrt{r}\cos\frac{\theta}{2}\cos\frac{x^4+\phi}{2}, & w^2 &= 2\sqrt{r}\cos\frac{\theta}{2}\sin\frac{x^4+\phi}{2}, \\ w^3 &= 2\sqrt{r}\sin\frac{\theta}{2}\cos\frac{x^4-\phi}{2}, & w^4 &= 2\sqrt{r}\sin\frac{\theta}{2}\sin\frac{x^4-\phi}{2}\,, \end{aligned} \tag{A.1}$$

then the solution given in (2.5) takes the form

$$\begin{aligned} dS^2 &= \psi(r)^{-1}\left[dx^+dx^- + (\psi(r)-1)(dx^+)^2\right] + \chi_i(w)\,dx^+\,dw^i + \widehat{g}_{mn}\,du^m du^n + \psi(r)dw^2, \\ x^\pm &\equiv x^5 \pm t\,, \quad r\equiv \frac{1}{4}w^2, \quad \psi(r)\equiv\left(1+\frac{r_0}{r}\right), \quad \chi_i(w)\,dw^i = \psi(r)^{-1}\,\frac{\widetilde{J}}{4r}\,(dx^4+\cos\theta\,d\phi) \\ C^{(2)} &= \frac{1}{2}C_{ij}(w)dw^i\wedge dw^j + C_{+i}(w)dx^+\wedge dw^i + C_{+-}(w)dx^+\wedge dx^-\,, \end{aligned} \tag{A.2}$$

where $C^{(2)}$ denotes the RR 2-form field and C_{ij}, C_{+i} and C_{+-} are some fixed functions of w. We can now use the following algorithm to generate the deformations describing left-moving transverse oscillations of the black hole:

1. We first consider a deformation of the solution generated by the diffeomorphism

$$\begin{aligned} w^i &\to w^i + a^i\,(x^+ + c)\,f + (x^+ + c)\,a\cdot w\, w^i\, g, \\ x^- &\to x^- - 2\,a\cdot w\,\psi^2\, f - (x^+ + c)\,\psi\,\left(a\cdot\chi\, f + a\cdot w\, w\cdot\chi\, g\right), \\ x^+ &\to x^+\,, \end{aligned} \tag{A.3}$$

where a denotes an arbitrary constant four dimensional vector, $a\cdot w \equiv a^i w^i$, c is an arbitrary constant and f and g are functions of r satisfying

$$g = \frac{1}{2}\,\psi^{-2}\,(\psi^2 f)'\,. \tag{A.4}$$

Here $'$ denotes derivative with respect to r. The diffeomorphism has been chosen such that all the terms in $\delta(dS^2)$ to first order in a^i are proportional to $(x^+ + c)$ without any derivative acting on it. By accompanying this diffeomorphism by a suitable gauge transformation of $C^{(2)}$ we can ensure that $\delta C^{(2)}$ also is proportional to $(x^+ + c)$ without any derivative acting on it.

2. We now replace the overall factor of $x^+ + c$ by an arbitrary function $\epsilon(x^+)$ everywhere in the deformed solution. Thus the deformed configuration is proportional to $\epsilon(x^+)$. Furthermore, by construction it is guaranteed to be a solution to the equations of motion for $\epsilon(x^+) = x^+ + c$. This in turn shows that if we substitute the deformed configuration into the equations of motion then the terms proportional to $\epsilon(x^+)$ and $\partial_+\epsilon(x^+)$ must vanish automatically.

3. Our goal is to ensure that the deformed configuration is a solution to the equations of motion to linear order in ϵ for arbitrary function $\epsilon(x^+)$. Since the field equations are second order in derivatives, and terms involving $\epsilon(x^+)$ and $\partial_+\epsilon(x^+)$ are guaranteed to vanish, it only remains to ensure that the terms involving $\partial_+^2\epsilon$ vanish. Such terms can arise in the $++$ component of the metric equation, and the vanishing of the term proportional to $\partial_+^2\epsilon$ can be shown to require[15]

$$G^{ij}\delta G_{ij} = 0\,, \tag{A.5}$$

where i, j run over the four transverse spatial coordinates, G_{ij} is the background metric and δG_{ij} denotes the first order deformation of the metric. This imposes one additional constraint on the functions f and g. Once this condition is satisfied we have a set of deformations parametrized by four arbitrary function $a^i\epsilon(x^+)$.[16]

At the end of the second step this procedure gives

$$\begin{aligned} \delta\left(dS^2\right) = &-\frac{1}{2}\,\epsilon(x^+)\,\psi^{-2}\psi'\,a\cdot w\,(f + 4rg)\,\left(dx^+\,dx^- - (dx^+)^2\right) \\ &+\frac{1}{2}\,\epsilon(x^+)\,\psi'\,a\cdot w\,(f + 4rg)\,dw^2 + \epsilon(x^+)\,\psi\,f'\,a\cdot dw\,w\cdot dw \end{aligned}$$

[15] Note that since the three form field strengths contain at most a single derivative of ϵ, they do not directly contribute any term proportional to $\partial_+^2\epsilon$ in the equations of motion.

[16] This analysis is similar in spirit, although much simpler than, the one carried out in [61].

$$
\begin{aligned}
&+2\,\epsilon(x^+)\,\psi\,g\,a\cdot dw\,w\cdot dw + 2\,\epsilon(x^+)\,\psi\,g\,a\cdot w\,dw^2 + \epsilon(x^+)\,\psi\,g'\,a\cdot w\,w\cdot dw\,w\cdot dw\\
&-\epsilon(x^+)\,\psi^{-1}\,dx^+\;d\left(\psi(a\cdot\chi\,f + a\cdot w\,w\cdot\chi\,g)\right)\\
&+\epsilon(x^+)\,\chi_i\,dx^+\,d\left(a^i\,f + a\cdot w\,w^i\,g\right) + \epsilon(x^+)\,\partial_k\chi^i\,(a^k\,f + a\cdot w\,w^k\,g)\,dx^+dw^i\,,\\
\delta\,C^{(2)} = {}&\frac{1}{2}\,\epsilon(x^+)\,(\partial_k\,C_{ij} + \partial_i\,C_{jk} + \partial_j\,C_{ki})\;(a^k f + a\cdot w\,w^k\,g)\,dw^i\wedge dw^j\\
&+\epsilon(x^+)\,\partial_k C_{+-}\,(a^k f + a\cdot w\,w^k\,g)\,dx^+\wedge dx^-\\
&+\epsilon(x^+)\,\partial_k C_{+-}\,dw^k\wedge\,(2\,\psi^2\,f\,a\cdot dw + a\cdot w\,(\psi^2 f)'\,w\cdot dw)\\
&-\epsilon(x^+)\,\partial_k\,C_{+-}\;(a\cdot\chi\,f + a\cdot w\,w\cdot\chi\,g)\;\psi\,dw^k\wedge dx^+\\
&+\epsilon(x^+)\,(\partial_l C_{+k} - \partial_k C_{+l})\,(a^l\,f + a\cdot w\,w^l\,g)\,dx^+\wedge dw^k\,. \qquad\qquad (A.6)
\end{aligned}
$$

Substituting this into (A.5) gives

$$2\psi'\,(f + 4rg) + \psi\,f' + 10\,\psi\,g + 4\,r\,\psi\,g' = 0\,. \qquad (A.7)$$

Using eq. (A.4) we can regard (A.7) as a second order linear differential equation for f. Thus it has two independent solutions. It is easy to verify that the general solution to (A.4), (A.7) is

$$f = (A_0\,r^{-2} + B_0)\,\psi^{-2}, \qquad g = -A_0\,r^{-3}\,\psi^{-2}\,, \qquad (A.8)$$

where A_0 and B_0 are two arbitrary constants. Requiring that the solution gives a normalizable deformation of the metric and the 2-form field near $r = 0$ we get $A_0 = 0$. Thus we have

$$f = B_0\,\psi^{-2}\,, \qquad g = 0\,. \qquad (A.9)$$

It is easy to verify that the deformations of the metric and the 2-form field associated with this choice of f is normalizable both at $r = 0$ and at $r = \infty$. Thus we have normalizable deformation of the solution parametrized by four indendent functions $a^i\,\epsilon(x^+)$. This shows the existence of four left-moving modes on the black hole world-volume. Furthermore the contribution to the norm of the deformation from the throat region $r \ll r_0$ vanishes, showing that these modes are located outside the horizon.

We expect that a similar argument can be used to construct the four left-moving fermionic modes on the black hole world-volume. In this case we shall need to use the broken supersymmetry generators to generate the fermionic deformation of the solution. However we shall not carry out this analysis explicitly.

B Explicit construction of the left-moving bosonic modes on the four dimensional black hole

In this appendix we shall give explicit construction of the bosonic zero modes living on the four dimensional black hole. We begin with the left-moving zero modes associated with the harmonic two form ω in the Taub-NUT space given in (3.11). For any 2-form field B — either the NSNS or RR sector 2-form field of the ten dimensional type IIB string theory or a four form field with two legs along an internal 2-cycle of $K3$ — we consider a deformation of the form

$$\delta B = \epsilon(x^+)\,\omega\,, \qquad (B.1)$$

for any function $\epsilon(x^+)$ of $x^+ = x^5 + t$. This gives

$$\begin{aligned} d(\delta\, B) &= \epsilon'(x^+)\, dx^+ \wedge \omega \\ &= -\epsilon'(x^+)\, \frac{1}{r^2\, R_4^2} \left(\frac{1}{r} + \frac{4}{R_4^2}\right)^{-2} \left(1 + \frac{r_0}{r}\right)^{-1} \\ &\quad \times (e_0 \wedge e_2 \wedge e_3 + e_0 \wedge e_4 \wedge e_5 + e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5)\,, \end{aligned} \tag{B.2}$$

where the 1-forms e_i's have been defined in (3.5). $d(\delta B)$ given in (B.2) can be shown to be anti-self-dual. Hence $d(\delta B)$ is both closed and co-closed and δB given in (B.1) provides a solution to the linearized equations of motion of $B_{\mu\nu}$ around the background (3.1). For the 3-form field strength deformation given in (B.2) one also finds that there is no contribution to the stress tensor from the interference term between the deformation and the leading order field strength given in (3.4). As a result the deformation (B.1) also satisfies the metric equation of motion at the linearized level. However in order that (B.1) corresponds to a valid configuration in string theory, B must correspond to a left-chiral 2-form (which has anti-self-dual field strength in our convention). Since type IIB on K3 has 2+19=21 left-chiral 2-form fields we get 21 left-moving bosonic modes from this construction. Finally this deformation is normalizable with the metric given in (3.1) and the norm is supported outside the throat, i.e. outside the $r \ll r_0, R_4^2$ region. Thus these modes should be counted as part of the black hole hair.

Next we shall describe the left-moving modes associated with the 3 transverse motion of the black hole. For this we introduce new coordinates (y^1, y^2, y^3) via

$$y^1 = r\, \cos\theta\, \cos\phi, \qquad y^2 = r\, \cos\theta\, \sin\phi, \qquad y^3 = r\, \cos\theta\,. \tag{B.3}$$

In this coordinate system the metric given in eq. (3.1) takes the form

$$\begin{aligned} dS^2 &= \psi(r)^{-1}\, \left\{dx^+ dx^- + (\psi(r) - 1)\, (dx^+)^2\right\} + \frac{\widetilde{J}}{4}\, \chi(r)\, \psi(r)^{-1}\, (dx^4 + A_\alpha(y)\, dy^\alpha)\, dx^+ \\ &\quad + \psi(r)\, \chi(r)^{-1}\, \left(dx^4 + A_\alpha(y)\, dy^\alpha\right)^2 + \psi(r)\, \chi(r)\, dy^2 + \widehat{g}_{mn}\, du^m du^n\,, \end{aligned} \tag{B.4}$$

where

$$\psi(r) = 1 + \frac{r_0}{r}\,, \quad \chi(r) = \frac{1}{r} + \frac{4}{R_4^2}\,, \quad A_\alpha(y)\, dy^\alpha = \cos\theta\, d\phi\,. \tag{B.5}$$

We can now generate an x^+ dependent deformation of this solution by first considering a diffeomorphism

$$\begin{aligned} y^\alpha &\to y^\alpha + (x^+ + c)\, (b^\alpha\, \widetilde{f} + b\cdot y\, y^\alpha\, \widetilde{g})\,, \\ x^- &\to x^- - 2\, b\cdot y\, \chi\, \psi^2\, \widetilde{f}\,, \\ x^4 &\to x^4 - (x^+ + c)\, A_\alpha\, (b^\alpha\, \widetilde{f} + b\cdot y\, y^\alpha\, \widetilde{g})\,, \end{aligned} \tag{B.6}$$

and then replacing $(x^+ + c)$ by $\epsilon(x^+)$ in the deformed solution. Here c, b^1, b^2, b^3 are arbitrary parameters, $b\cdot y \equiv b^\alpha\, y^\alpha$, and $\widetilde{f}$ and $\widetilde{g}$ are functions satisfying

$$\widetilde{g} = \frac{1}{r}\, \psi^{-2}\, \chi^{-1} (\psi^2\, \chi\, \widetilde{f})'\,. \tag{B.7}$$

JHEP07(2009)091

This gives

$$
\begin{aligned}
\delta(dS^2) \;=\; & -\epsilon(x^+)\,\psi^{-2}\,\psi'\,\frac{b\cdot y}{r}\,(\tilde f + r^2\tilde g)\,\left(dx^+dx^- - (dx^+)^2\right) \\
& + \frac{\widetilde J}{4}\,\epsilon(x^+)\,(\psi^{-1}\chi)'\,\frac{b\cdot y}{r}\,(\tilde f + r^2\tilde g)\,dx^+\,(dx^4 + A\cdot dy) \\
& + \frac{\widetilde J}{4}\,\epsilon(x^+)\,\psi^{-1}\,\chi\,(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\,(b^\alpha\,\widetilde f + b\cdot y\,y^\alpha\,\widetilde g)\,dy^\beta\,dx^+ \\
& + \epsilon(x^+)\,(\psi\,\chi^{-1})'\,\frac{b\cdot y}{r}\,(\tilde f + r^2\tilde g)\,(dx^4 + A\cdot dy)^2 \\
& + 2\,\epsilon(x^+)\,\psi\,\chi^{-1}\,(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\,(b^\alpha\,\widetilde f + b\cdot y\,y^\alpha\,\widetilde g)\,dy^\beta\,(dx^4 + A\cdot dy) \\
& + \epsilon(x^+)\,(\psi\,\chi)'\,\frac{b\cdot y}{r}\,(\tilde f + r^2\tilde g)\,dy^2 + 2\,\epsilon(x^+)\,\psi\,\chi\,dy^\alpha\,d\left(b^\alpha\,\widetilde f + b\cdot y\,y^\alpha\,\widetilde g\right)\,. \qquad \text{(B.8)}
\end{aligned}
$$

One can construct the deformation of the 2-form field in a straightforward manner but we shall not do this here.[17] Our construction guarantees that when we substitute the deformation (B.8) (and the corresponding deformation of the 2-form field) into the linearized equations of motion in the black hole background, all terms up to first derivative of $\epsilon(x^+)$ vanish. Requiring the coefficient of the $\partial_+^2\epsilon$ term to vanish gives us the equation:

$$
\psi^{-1}\,\chi\,(\psi\,\chi^{-1})'(\widetilde f + r^2\,\widetilde g) + 3\,\psi^{-1}\,\chi^{-1}\,(\psi\,\chi)'(\widetilde f + r^2\,\widetilde g) + 2\,(\widetilde f' + 4\,r\,\widetilde g + r^2\,\widetilde g') = 0\,. \qquad \text{(B.9)}
$$

Using eq. (B.7) we can regard (B.9) as a second order linear differential equation for $\widetilde f$. Thus it has two independent solutions. It is easy to verify that the general solution to (B.7), (B.9) is

$$
\widetilde f = (\widetilde A_0\,r^{-3} + \widetilde B_0)\,\psi^{-2}\,\chi^{-1}, \qquad \widetilde g = -3\,\widetilde A_0\,r^{-5}\,\psi^{-2}\,\chi^{-1}\,, \qquad \text{(B.10)}
$$

where $\widetilde A_0$ and $\widetilde B_0$ are two arbitrary constants. Requiring that the solution gives a normalizable deformation of the metric and the 2-form field near $r = 0$ we get $\widetilde A_0 = 0$. Thus we have

$$
\widetilde f = \widetilde B_0\,\psi^{-2}\,\chi^{-1}, \qquad \widetilde g = 0\,. \qquad \text{(B.11)}
$$

It is easy to verify that the deformations of the metric and the 2-form field associated with this choice of $\widetilde f$ is normalizable both at $r = 0$ and at $r = \infty$. Thus we have normalizable deformation of the solution parametrized by three indendent functions $b^\alpha\,\epsilon(x^+)$. This shows the existence of three left-moving modes on the black hole world-volume describing the left-moving transverse oscillation modes of the black hole. Furthermore the contribution to the norm of the deformation from the throat region $r \ll r_0, R_4^2$ vanishes, showing that these modes are located outside the horizon.

Finally we turn to the zero modes describing the motion of the D1-D5 system relative to the Taub-NUT space. We shall not carry out the construction in detail but describe these deformations in the limit $R_4^2 \gg r_0$. To leading order in this limit, the deformations

[17] For this one needs to accompany the diffeomorphism (B.6) by an appropriate gauge transformation of the 2-form field such that every term in the deformation has an explicit factor of $(x^+ + c)$ without any derivative acting on it. We then replace $(x^+ + c)$ by $\epsilon(x^+)$.

associated with these left-moving modes are in fact given by the ones described in (A.6). Indeed the arguments of appendix A show that for $r \ll R_4^2$ when the Taub-NUT metric can be replaced by flat metric, the deformations given in (A.6) satisfy the linearized equations of motion. On the other hand since the function f in (A.6) approaches a constant for $r \gg r_0$, the metric fluctuations fall off as $1/r^2$ and the contribution to the norm of the deformation from this region is small. Thus the deformation given in (A.6) is supported in the region $r \sim r_0$, and for $r \sim R_4^2 \gg r_0$, where the deviation of the Taub-NUT metric from the flat metric becomes significant, the deformation is close to zero. Thus we conclude that in the region where the deformation (A.6) is supported it remains an approximate solution to the equations of motion.[18]

Our analysis also allows us to determine the J quantum numbers of various deformations. Since in the region $r \ll R_4^2$ the parameters a labelling the deformation in (A.6) transform in the vector representation of the SO(4) rotation group acting on the coordinates w, they carry $J = \pm 1$. This may also be seen by noting that under a translation $x^4 \to x^4 + \beta$, these modes transform with a phase $e^{\pm i\beta/2}$. Since x^4 has period 4π, this shows that these modes carry ± 1 quantum of x^4 momentum. On the other hand the deformations describing the overall transverse motion of the black hole, described by the parameters b^α, are neutral under x^4 translation, and hence has $J = 0$. The different transformation properties of the modes labelled by a and b help demonstrate that they are distinct deformations of the solution.

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JHEP07(2009)091